Superluminality in the Fierz–Pauli massive gravity

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Abstract

We study the propagation of helicity-1 gravitons in the Fierz–Pauli massive gravity in nearly Minkowski backgrounds. We show that, generically, there exist backgrounds consistent with field equations, in which the propagation is superluminal. The relevant distances are much longer than the ultraviolet cutoff length inherent in the Fierz–Pauli gravity, so superluminality occurs within the domain of validity of the effective low energy theory. There remains a possibility that one may get rid of this property by imposing fine tuning relations between the coefficients in the non-linear generalization of the Fierz–Pauli mass term, order by order in non-linearity; however, these relations are not protected by any obvious symmetry. Thus, among others, superluminality is a problematic property to worry about when attempting to construct infrared modifications of General Relativity.

1 Introduction and summary

Lorentz-invariant massive gravity in four dimensions — the Fierz–Pauli theory — is of interest from the viewpoint of understanding the problems that may arise when one attempts to modify General Relativity in the infrared domain. Indeed, at the linearized level about Minkowski background, classical Fierz–Pauli theory exhibits the van Dam–Veltman–Zakharov discontinuity [1, 2] and non-linearity at large distances from gravitating bodies [3], whereas at the quantum level this theory becomes strongly coupled at unacceptably low energies [4]. In slightly curved backgrounds, the Fierz–Pauli gravity and its non-linear generalizations necessarily contain an extra, Boulware–Deser mode [5], over and beyond the five modes of massive graviton; furhermore, one of the modes is always a ghost. These properties of the Fierz–Pauli massive gravity are reviewed, e.g., in Ref. [6].

In this note we point out that the Fierz–Pauli theory has yet another peculiar feature. Namely, we find that, generically, there exist slighly curved backgrounds consistent with field equations, in which some of helicity-1 graviton modes are superluminal. We recall that these modes are not pathological otherwise: they do not exhibit the vDVZ discontinuity, Vainshtein non-linearity or particularly bad UV behavior at the quantum level[1]. We check that the superluminal propagation occurs over large distances as compared to the UV scale intrinsic in the Fierz–Pauli gravity, so this phenomenon shows up within the domain of applicability of the effective low energy theory.

In this regard, the Fierz–Pauli massive gravity is analogous to the Dvali–Gabadadze–Porrati model [7], which also has a superluminal mode in some legitimate backgrounds [8]. As discussed in Ref. [8], superluminal propagation signals that there is no UV completion of an effective low energy theory into quantum field theory or perturbative string theory, so the property we have observed is yet another feature to worry about.

Our analysis has a loophole, though. There remains a possibility that the superluminal propagation of helicity-1 modes may be avoided by imposing an infinite set of fine tuning relations on the parameters in the non-linear generalization of the Fierz–Pauli mass term, order by order in the degree of non-linearity. These relations, however, are not protected by any obvious symmetry, so we do not find this "solution" attractive.

It is worth stressing that we study here massive gravities that have the Minkowski metric as a solution to the field equations. The situation may be less pathological in theories whose "natural" background is different from Minkowski; a well studied example is massive gravity about (anti) de Sitter space-time [9, 10, 11, 12, 13, 14].

Once Minkowski space-time is a solution to the field equations, the general mass term is a polynomial in $(g_{\mu\nu} - \eta_{\mu\nu})$. The original Fierz–Pauli action is quadratic,

$$S_{FP} = \frac{m_G^2}{64\pi G} \int d^4x \left\{ -\eta^{\mu\lambda}\eta^{\nu\rho}(g_{\mu\nu} - \eta_{\mu\nu})(g_{\lambda\rho} - \eta_{\lambda\rho}) + \left[\eta^{\mu\nu}(g_{\mu\nu} - \eta_{\mu\nu})\right]^2 \right\} \ . \tag{1}$$

Our purpose is to evaluate the speed of helicity-1 modes of metric perturbations in nearly Minkowski backgrounds. So, we will make use of the perturbation theory in $(\overline{g}_{\mu\nu} - \eta_{\mu\nu})$, where $\overline{g}_{\mu\nu}$ is the background metric. To this end, we will have to generalize the above expression by adding cubic, and then higher-order terms. It is at this stage that the fine tuning "solution" of the superluminality problem shows up.

We begin with studying in section 2 the Fierz–Pauli theory and its generalizations in cosmological backgrounds of small space-time curvature. In these backgrounds, the metric perturbations decompose into the tensor, vector and scalar modes with respect to spatial rotations. To set the stage, we consider in section 2.1 a particular form of the mass term

[1] Helicity-0 (longitudinal) modes may be superluminal too, but they are less interesting in our context because of the other pathologies inherent in the longitudinal sector.

(1) and find that the vector perturbations in this case are superluminal in fairly general backgrounds. By itself, this is not a particularly strong result, however. The degree of superluminality (c^2-1), where c is the speed of the propagation of the vector modes of metric perturbations, is proportional to the deviation of the background from Minkowski space-time, $(\overline{g}_{\mu\nu}-\eta_{\mu\nu})$. This implies that the contributions to (c^2-1) due to $\mathcal{O}\left[(g_{\mu\nu}-\eta_{\mu\nu})^3\right]$-terms in the action are of the same order. We consider the cubic terms in section 2.2 and find that to the first order in $(\overline{g}_{\mu\nu}-\eta_{\mu\nu})$, the total (c^2-1) may be made equal to zero by fine tuning a parameter in the cubic action. To see that similar fine tuning is required in higher orders in $(\overline{g}_{\mu\nu}-\eta_{\mu\nu})$, we have to study the action containing higher orders in $(g_{\mu\nu}-\eta_{\mu\nu})$.

Proceeding with cosmological backgrounds is technically challenging at this stage. Instead, we make use of the Stückelberg formalism. We consider backgrounds of a special form and gravitons traveling along a particular direction. This enables us to separate helicity-1 modes from other metric perturbations in a consistent way. In section 3.1 we cross check by rederiving, within this formalism, the fine tuning relation ensuring that $c^2=1$ at the linear order in $(\overline{g}_{\mu\nu}-\eta_{\mu\nu})$. In section 3.2 we show explicitly that for the backgrounds we consider, the requirement that helicity-1 modes do not propagate in superluminal way gives rise, at the quadratic order in $(\overline{g}_{\mu\nu}-\eta_{\mu\nu})$, to further fine tuning relations, now involving the coefficients in the fourth-order action. It is then straightforward to convince oneself that fine tuning proliferates to higher orders. We have found no arguments suggesting that our fine tuning relations are sufficient to avoid superluminal propagation of helicity-1 modes in arbitrary backgrounds; neither have we found backgrounds that would rule out the fine-tuning "solution" to the problem of superluminality in the helicity-1 sector.

2 Cosmological backgrounds

The theory we consider in this paper is the Fierz-Pauli model whose action is

$$S=S_{GR}+S_m$$

where S_m is the mass term including the quadratic part (1) and possible higher order terms, while S_{GR} is the action of General Relativity. In the further analysis we set $16\pi G=1$. Our purpose in this section is to study the propagation of gravitons in the spatially flat cosmological backgrounds, in the regime when spatial momenta and frequencies are much higher than the graviton mass. In the coordinate frame where $\eta_{\mu\nu}$ in the mass term is $\mathrm{diag}[1,-1,-1,-1]$, the spatially flat FRW metric is

$$ds^2=n^2(t)a^2(t)dt^2-a^2(t)d\mathbf{x}^2 \tag{2}$$

In what follows, it is convenient to use conformal time η related to time t by

$$d\eta=n(t)dt$$

In the coordinates $(\eta, \mathbf{x})$, the speed of light is equal to 1. We will consider nearly flat metric, for which $\delta n = n - 1$ and $\delta a = a - 1$ are small.

The field equations for the background, written in terms of conformal time, read

$$\left(\frac{a'}{a}\right)^2 \equiv \mathcal{H}^2 = \epsilon_0, \quad 2\frac{a''}{a} - \mathcal{H}^2 = \epsilon_s \tag{3}$$

where prime denotes $d/d\eta$, the contributions ϵ_0 , ϵ_s are due to the mass term and are of order

$$\epsilon_0 \ , \ \epsilon_s = m_G^2 \left[\mathcal{O}(a-1) + \mathcal{O}(n-1)\right]$$

The lowest order terms come from the quadratic part (1); their explicit expressions are

$$\epsilon_0 = -\frac{m_G^2 n}{2}(a^2 - 1), \quad \epsilon_s = -\frac{m_G^2}{2n}(2(a^2-1) + (a^2 n^2 - 1)) \tag{4}$$

where the expansion up to the first order in δa and δn is understood.

Now, we can freely choose δn and δa at a given moment of time. Then these variables will change in time at low rate determined by the value of m_G, which we take small. We will consider the time scales shorter than m_G^{-1}, so we treat δn and δa as constants. It is worth noting that the consistency of the system (3), i.e., covariant conservation of the effective energy-momentum tensor coming from the mass terms, implies that $n(\eta)$ obeys the equation

$$\frac{\partial \epsilon_0}{\partial n} n' = -\frac{\partial \epsilon_0}{\partial a} a' + \mathcal{H}(\epsilon_s - \epsilon_0) \ . \tag{5}$$

However, this equation determines the evolution of δn and does not prohibit one to choose arbitrarily the value of δn at a given moment of time. So, for our purposes we can indeed treat δn and δa as independent free parameters of the background.

Let us now include metric perturbations, and write the perturbed metric in the following form,

$$ds^2 = a^2(t)\left[n^2(t)(1 + h_{00})dt^2 + 2h_{0i}n(t)dtdx^i + (-\delta_{ik} + h_{ik})dx^i dx^k\right] \ .$$

Due to the $O(3)$ symmetry of the FRW metric one can study different helicity sectors separately. We will be interested in vector (helicity-1) perturbations, for which, using the standard notation, we write

$$h_{ij} = \partial_i W_j + \partial_j W_i, \quad \partial_i W_i = 0,$$
$$h_{00} = 0, \quad h_{0i} = U_i, \quad \partial_i U_i = 0$$

The question is whether or not the propagating vector modes are superluminal for some choice of δa and δn.

2.1 Quadratic mass term only

To warm up, let us consider the particular form of the graviton mass term given by (1), without cubic term. The complete quadratic action for vector perturbations in the background obeying the equations of motion (3) is

$$
\begin{aligned}
S_{tot}^{(2)}\Big|_{E.O.M.} = \quad & \frac{1}{2}\int a^2 d\eta d^3x \Big[\ 2U_i\Delta W_i' - U_i\Delta U_i + W_i''\Delta W_i + 2\mathcal{H}W_i'\Delta W_i + 6\epsilon_0 U_iU_i + 2\epsilon_s W_i\Delta W_i\ \Big] \\
& + \int d^3x dt \left(\frac{\delta m^2}{2}U_iU_i + \frac{\delta M^2}{2}W_i\Delta W_i\right)
\end{aligned}
$$

where $\Delta = \partial_i\partial_i$. The last term in the above expression comes directly from the mass term in the action. In the case (1) the explicit expressions for δm^2, δM^2 to the first order in δn, δa are

$$
\delta m^2 = m_G^2(1 + 4\delta a + 2\delta n), \quad \delta M^2 = m_G^2(1 + 4\delta a)
$$

It is convenient to define

$$
M^2 \equiv 4\epsilon_s + \frac{2}{a^2 n}\delta M^2, \quad m^2 \equiv 12\epsilon_0 + \frac{2}{a^2 n}\delta m^2
$$

Then the linear equations for metric perturbations are

$$
\begin{cases}
M^2\Delta W - 4\mathcal{H}\Delta U - 2\Delta U' + 2\Delta W'' + 4\mathcal{H}\Delta W' = 0 \\
m^2 U + 2\Delta W' - 2\Delta U = 0
\end{cases}
$$

We now concentrate on frequency and momenta exceeding H_0 and m_G, and make the Fourier transformation $\frac{\partial}{\partial\eta} \to i\omega,\ \Delta \to -p^2$. In this limit we obtain the following dispersion relation for the vector perturbations,

$$
c^2 \equiv \frac{\omega^2}{p^2} = \frac{M^2}{m^2} \tag{6}
$$

To the first order in δn, δa we find

$$
c^2 - 1 = -4\delta n \tag{7}
$$

This implies that for any background with $\delta n < 0$, the speed of vector perturbations exceeds the speed of light.

Importantly, $c^2 - 1$ as given by (7) is of the zeroth order in m_G^2. On the other hand, $\delta n'$ is of the first order in this parameter. So, at small m_G^2 there is enough time for the vector perturbations to propagate, in superluminal way, before the space-time metric changes. In detail, the metric does not change during the time interval $\tau \sim m_G^{-1}\cdot|\delta a|^{1/2}$, where the factor $|\delta a|^{1/2}$ is obtained by inspecting eqs. (4) and (5). The distance the vector perturbations advance light in this time interval is $L = \delta c\cdot\tau \sim m_G^{-1}\cdot|\delta a|^{1/2}|\delta n|$, which for not very small

δa and δn is much greater than the strong coupling scale [4] inherent in the Fierz–Pauli theory, $\Lambda_{UV}^{-1} = (m_G^4 M_{Pl})^{-1/5}$. Hence, the superluminal propagation is not hidden below the UV distance scale. This reasoning applies word by word to massive gravities with more general actions, to be discussed below. The effect of the superluminal propagation we analyze in this paper does occur within the domain of applicability of the effective field theory.

2.2 The cubic action

To the linear order in δn, δa, the speed of vector perturbations may receive contribution from cubic in $(g_{\mu\nu} - \eta_{\mu\nu})$ part of the mass term. One may thus wonder whether superluminality may be eliminated by the proper choice of the parameters in this part of the action. To examine this issue, let us calculate the linear in δa, δn contribution to c^2 for general cubic term. As before, the mass term in the action is assumed to be Lorentz-invariant, contain no derivatives and have flat metric as a solution to the massive gravity equations. Its quadratic part is of the Fierz–Pauli structure. The general expression for the cubic part is

$$S_m^{(3)} = \int d^4x \ \left(\mathcal{C} \cdot A_1^3 + \mathcal{E} \cdot A_2 A_1 + \mathcal{F} \cdot A_3 \right) \ , \tag{8}$$

where

$$\begin{aligned} A_1 &= \eta^{\mu\nu}(g_{\mu\nu} - \eta_{\mu\nu}), \quad A_2 = \eta^{\mu\lambda}\eta^{\nu\rho}(g_{\mu\nu} - \eta_{\mu\nu})(g_{\lambda\rho} - \eta_{\lambda\rho}) \ , \\ & A_3 = \eta^{\mu\lambda}\eta^{\rho\sigma}\eta^{\nu\tau}(g_{\mu\nu} - \eta_{\mu\nu})(g_{\lambda\rho} - \eta_{\lambda\rho})(g_{\sigma\tau} - \eta_{\sigma\tau}) \end{aligned} \tag{9}$$

with $\mathcal{C}$, $\mathcal{E}$, $\mathcal{F}$ being arbitrary coefficients. The total mass part of the action is the Fierz-Pauli term (1) plus $S_m^{(3)}$.

The expressions (3) – (6) are still valid, but δm and δM are to be recalculated. The linearized expressions are

$$\begin{aligned} \delta m^2 &= m_G^2(1 + 4\delta a + 2\delta n) - 8\mathcal{E}(4\delta a + \delta n) - 12\mathcal{F}(2\delta a + \delta n) \\ \delta M^2 &= m_G^2(1 + 4\delta a) - 8\mathcal{E}(4\delta a + \delta n) - 24\mathcal{F}\delta a \end{aligned}$$

The previous result (7) then modifies to

$$c^2 - 1 = \frac{3\mathcal{F} - m_G^2}{m_G^2} \cdot 4\delta n \tag{10}$$

Thus, to avoid superluminality at the first order in δa, δn, one has to impose the fine tuning relation

$$3\mathcal{F} = m_G^2 \tag{11}$$

We will reproduce this fine tuning relation in section 3.1 in the Stückelberg approach, and in section 3.2 we proceed to higher order terms.

3 The Stückelberg approach

3.1 Cubic order fine tuning revisited

To continue the analysis, it is convenient to use the Stückelberg formalism, wich will also allow us to study backgrounds other than cosmological. Let us write the metric as follows,

$$g_{\mu\nu} = \eta_{\mu\nu} + \overline{h}_{\mu\nu} + h_{\mu\nu} \ , \tag{12}$$

where the first term is the Minkowski metric, the second term corresponds to non-trivial background and is assumed to be small, and the third one describes perturbations about this background.

To perform the Stückelberg analysis, one enlarges the set of fields in the theory by introducing new fields ξ^μ and $\tilde{g}_{\mu\nu}$ in the way dictated by the gauge symmetry of General Relativity[4, 15, 16, 17], see also Ref. [6] for detailed discussion,

$$g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x+\xi) + \partial_\mu \xi^\lambda \, \tilde{g}_{\nu\lambda}(x+\xi) + \partial_\nu \xi^\lambda \, \tilde{g}_{\mu\lambda}(x+\xi) + \partial_\mu \xi^\lambda \, \partial_\nu \xi^\rho \, \tilde{g}_{\lambda\rho}(x+\xi) \ .$$

We are going to consider high graviton momenta and slowly varying backgrounds, so we neglect the derivatives of $\overline{h}_{\mu\nu}$. Once the field $\tilde{g}_{\mu\nu}(x)$ is gauge fixed, mixing between its fluctuations and the field ξ^μ becomes irrelevant at high enough momenta, and we are left with the theory of one vector field ξ^μ determining the interesting part of metric perturbations via

$$h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_\mu \xi^\lambda \overline{h}_{\lambda\nu} + \partial_\nu \xi^\lambda \overline{h}_{\lambda\mu} + \partial_\mu \xi^\lambda \partial_\nu \xi_\lambda + \overline{h}_{\lambda\rho} \partial_\mu \xi^\lambda \partial_\nu \xi^\rho \ . \tag{13}$$

Throughout this paper indices are raised and lowered with the Minkowski metric. Note that, nevertheless, this expression is exact for slowly varying backgrounds, i.e., no expansion in $\overline{h}_{\mu\nu}$ has been made yet.

The field ξ^μ does not enter the Einstein–Hilbert part of the action, and obtains kinetic term from the mass term in the original action. To evaluate its speed, we study the action to the quadratic order in ξ^μ. We begin with the linear order in $\overline{h}_{\mu\nu}$; in this way we are going to reproduce the fine tuning relation (11) in the Stückelberg formalism. Hence, we need the mass term to the cubic order in $(g_{\mu\nu} - \eta_{\mu\nu})$,

$$S_m^{(2+3)} = S_{FP} + S_m^{(3)} = \int d^4x \left[\mathcal{A} \cdot (-A_1^2 + A_2) + \mathcal{C} \cdot A_1^3 + \mathcal{E} \cdot A_2 A_1 + \mathcal{F} \cdot A_3 \right] \ ,$$

where

$$\mathcal{A} \equiv -m_G^2/4 \ , \tag{14}$$

and the combinations A_1, A_2, A_3 are given by (9). According to the above discussion, we will need the terms in this action that are quadratic in the Stückelberg field ξ^μ and zero and

first order in $\overline{h}_{\mu\nu}$. Note that these terms appear both due to the quadratic and cubic terms in the action and due to the non-linearity of the gauge transformation (13).

Making use of (12) and (13) one obtains $S_m^{(2+3)} = \int\ d^4x\ \mathcal{L}_m^{(2+3)}$, with

$$\begin{aligned}\mathcal{L}_m^{(2+3)}(\overline{h},\xi) = \quad & \Big\{-2\mathcal{A}\left[(\partial_\mu\xi^\mu)^2 + \xi_\mu\Box\xi^\mu\right] - (2\mathcal{A}+3\mathcal{F})\,\xi_\lambda\overline{h}^{\mu\nu}\partial_\mu\partial_\nu\xi^\lambda \\ & +2(4\mathcal{E}+3\mathcal{F}-2\mathcal{A})\,\partial_\lambda\xi^\lambda\cdot\overline{h}^{\mu\nu}\partial_\mu\xi_\nu \\ & +2(\mathcal{A}-\mathcal{E})\,\overline{h}\xi_\mu\Box\xi^\mu + 2(\mathcal{E}+6\mathcal{C})\,\overline{h}(\partial_\mu\xi^\mu)^2 - (3\mathcal{F}+4\mathcal{A})\,\overline{h}_{\mu\nu}\xi^\mu\Box\xi^\nu\Big\}\,,\end{aligned} \tag{15}$$

where

$$\overline{h} \equiv \overline{h}_{\mu\nu}\eta^{\mu\nu}\ .$$

Let us consider the frame in which the Minkowski metric entering the mass terms has the standard form $\eta_{\mu\nu} = \text{diag}[1,-1,-1,-1]$ and study the backgrounds whose space-time metric is diagonal in this frame, $\overline{h}_{\mu\nu} = \text{diag}[\overline{h}_{00},\overline{h}_{11},\overline{h}_{22},\overline{h}_{33}]$, so that

$$ds^2 = (1+\overline{h}_{00})dt^2\ +\ \sum_i(-1+\overline{h}_{ii})(dx^i)^2\ . \tag{16}$$

For time-independent $\overline{h}_{\mu\nu}$ this metric is a solution to the Einstein equations, so in massive theory the derivatives of $\overline{h}_{\mu\nu}$ are proportional to m_G^2. Hence, it is legitimate to consider this metric as slowly varying. This justifies the use of the expression (13), and also enables us to neglect the derivatives of $\overline{h}_{\mu\nu}$ when solving the equations for ξ^μ.

Let us further specify to gravitons propagating along the 1-st axis, whose momenta are

$$P_0 = \omega, \quad P_i = p\delta_{i\,1}$$

In this case the action (15) is a sum of three independent parts: the action that involves the field ξ^2 only, the action for ξ^3 and the action for the pair (ξ^0,ξ^1). An obvious reason for this decoupling is that the matrix $\overline{h}_{\mu\nu}$ does not mix, say, ξ^2 with other components of ξ^μ, while $\partial_\mu\xi^\mu$ involves ξ^0 and ξ^1 only. Hence, the transverse polarizations ξ^2 and ξ^3 decouple from each other and from longitudinal polarizations. These are precisely the (linear combinations of) helicity ±1 modes we are interested in. Their dispersion relations are readily calculated. One finds that the dispersion relation for both ξ^2 and ξ^3 to the linear order in $\overline{h}_{\mu\nu}$ is

$$\omega^2 = \left[1 - \frac{2\mathcal{A}+3\mathcal{F}}{2\mathcal{A}}(\overline{h}_{00}+\overline{h}_{11})\right]\cdot p^2\ .$$

The physical speed of these modes (recall that they propagate along the 1-st axis) is

$$c^2 = \frac{1}{n^2}\frac{\omega^2}{p^2}\ , \tag{17}$$

where

$$n^2 \equiv \frac{g_{00}}{|g_{11}|} = \frac{1+\overline{h}_{00}}{1-\overline{h}_{11}} \tag{18}$$

Hence to the linear order in $\overline{h}_{\mu\nu}$, the physical speed of helicity-1 gravitons is given by

$$c^2 = 1 - \frac{(4\mathcal{A}+3\mathcal{F})}{2\mathcal{A}}(\overline{h}_{00}+\overline{h}_{11}) \tag{19}$$

where $(\overline{h}_{00}+\overline{h}_{11})$ may have either sign. Recalling (14) and having in mind the two forms of background metric (2) and (16), we see that the expression (19) coincides with (10). In this way we recover the result of section 2.2: the condition for the absence of superluminal propagation of helicity-1 modes at the linear order in $\overline{h}_{\mu\nu}$ is $3\mathcal{F} = -4\mathcal{A} \equiv m_G^2$.

3.2 Fine tuning at fourth order and beyond

Let us now study the theory tuned according to (11). At the first order in $\overline{h}_{\mu\nu}$, helicity-1 gravitons in this theory propagate at high enough momenta precisely with the speed of light. We will now see that at the second order, superluminlity of helicity-1 gravitons generically reappears, unless one imposes additional fine tuning relations.

Let us introduce the general fourth-order mass terms and expand the action up to the order of $\mathcal{O}(\overline{h}^2 \times h^2)$. The fourth order terms are

$$S_m^{(4)} = \int d^4x \, \left[\mathcal{I}\cdot(A_4) + \mathcal{J}\cdot(A_1)^4 + \mathcal{K}\cdot(A_1A_3) + \mathcal{L}\cdot(A_1^2A_2) + \mathcal{M}(A_2)^2\right] .$$

Here, in analogy to $A_{2,3}$ in (8), the irreducible term is

$$A_4 = \eta^{\mu_2\nu_1}\dots\eta^{\mu_1\nu_4}\cdot(g_{\mu_1\nu_1}-\eta_{\mu_1\nu_1})\dots(g_{\mu_4\nu_4}-\eta_{\mu_4\nu_4})$$

and $\mathcal{I},\dots\mathcal{M}$ are arbitrary constants. Plugging the Stückelberg decomposition (13) into this action, and keeping the terms of order $\mathcal{O}(\overline{h}^2 \times h^2)$ in the complete action, we obtain the contribution to the quadratic action of the Stückelberg fields in the form $S_m^{(4)} = \int d^4x \, \mathcal{L}^{(4)}$, where

$$\begin{aligned}
\mathcal{L}_m^{(4)}(\overline{h},\xi) = &- (3\mathcal{C}+2\mathcal{L})\,(\overline{h}^2\xi^\mu\Box\xi_\mu) - (\mathcal{E}+4\mathcal{M})\,(\overline{h}_{\nu\rho}\overline{h}^{\nu\rho}\cdot\xi^\mu\Box\xi_\mu) - (2\mathcal{E}+3\mathcal{K})\,(\overline{h}\xi^\lambda\overline{h}^{\mu\nu}\partial_\mu\partial_\nu\xi_\lambda)\\
&- (3\mathcal{F}+4\mathcal{I})\,(\xi^\lambda\overline{h}^{\mu\rho}\overline{h}_\rho^\nu\partial_\mu\partial_\nu\xi_\lambda) - (2\mathcal{A}+6\mathcal{F}+4\mathcal{I})\,(\overline{h}_{\mu\rho}\overline{h}^{\rho\nu}\xi^\mu\Box\xi_\nu)\\
&+ 2(3\mathcal{F}+4\mathcal{E}+4\mathcal{I}+6\mathcal{K})\,(\partial_\lambda\xi^\lambda\cdot\overline{h}_\nu^\mu\overline{h}^{\nu\rho}\partial_\mu\xi_\rho) + 2(12\mathcal{J}+\mathcal{L})\,\overline{h}^2(\partial_\mu\xi^\mu)^2\\
&- (3\mathcal{K}+4\mathcal{E}-2\mathcal{A})\,(\overline{h}\overline{h}_{\mu\nu}\xi^\mu\Box\xi^\nu) + (6\mathcal{K}+16\mathcal{L}+24\mathcal{C}+4\mathcal{E})\,(\overline{h}\partial_\lambda\xi^\lambda\cdot\overline{h}^{\mu\nu}\partial_\mu\xi_\nu)\\
&+ 2(-\mathcal{A}+3\mathcal{F}+4\mathcal{E}+8\mathcal{M}+2\mathcal{I})\,(\overline{h}^{\mu\nu}\partial_\mu\xi_\nu)^2 - (2\mathcal{A}+6\mathcal{F}+4\mathcal{I})\,(\xi^\rho\overline{h}^{\mu\nu}\overline{h}_{\rho\lambda}\partial_\mu\partial_\nu\xi^\lambda)
\end{aligned}$$

The total Lagrangian is the sum of this term and (15). We continue to consider the diagonal background metric (16) and gravitons propagating along the 1-st axis. The fields ξ^2 and ξ^3

again decouple, for the same reason as in section 3.1. Making use of (11), we obtain the following dispersion relation for the coordinate frequency of the mode ξ^2 to the second order in $\overline{h}_{\mu\nu}$

$$\begin{aligned}\omega^2 = \quad & \Big\{1 + (\overline{h}_{00} + \overline{h}_{11}) \\ & + \left[\overline{h}_{00} + \frac{3\mathcal{A} - 2\mathcal{I}}{\mathcal{A}}\overline{h}_{22} + \frac{2\mathcal{A} - 4\mathcal{E} - 3\mathcal{K}}{2\mathcal{A}}\overline{h} + 2\frac{\mathcal{A} - \mathcal{I}}{\mathcal{A}}(\overline{h}_{00} - \overline{h}_{11})\right](\overline{h}_{00} + \overline{h}_{11})\Big\} \cdot p^2 \,.\end{aligned}$$

Finally, expanding n^{-2}, defined according to (18), to the second order in $\overline{h}_{\mu\nu}$ we find that the physical speed (17) of the helicity-1 graviton with the polarization ξ^2 is given by

$$c^2 = 1 + \left[\frac{3\mathcal{A} - 2\mathcal{I}}{\mathcal{A}}\overline{h}_{22} + \frac{2\mathcal{A} - 4\mathcal{E} - 3\mathcal{K}}{2\mathcal{A}}\overline{h} + \frac{3\mathcal{A} - 2\mathcal{I}}{\mathcal{A}}(\overline{h}_{00} - \overline{h}_{11})\right](\overline{h}_{00} + \overline{h}_{11})$$

Since the quantities $(\overline{h}_{00} + \overline{h}_{11})$, $(\overline{h}_{00} - \overline{h}_{11})$, $\overline{h}_{22}$ and $\overline{h}$ are independent of each other, we see that to avoid the superluminal propagation, one has to impose additional fine tuning relations

$$3\mathcal{A} - 2\mathcal{I} = 0 \,, \qquad 2\mathcal{A} - 4\mathcal{E} - 3\mathcal{K} = 0 \,. \tag{20}$$

The same relations ensure that the mode ξ^3 is not superluminal in the background we consider.

The analysis in this section reveals the following property: the dispersion relation for helicity-1 gravitons involves the coefficients of the irreducible higher order terms, which are multiplied by sign-indefinite combinations of the background metric. At the third order, the relevant coefficient is $\mathcal{F}$, the irreducible term is A_3, and the combination of metric is $(\overline{h}_{00} + \overline{h}_{11})$, while at the fourth order, these are $\mathcal{I}$, A_4, and $\overline{h}_{22}(\overline{h}_{00} + \overline{h}_{11})$, respectively. Unless these coefficients are fine tuned, helicity-1 gravitons propagate in superluminal way in backgrounds with appropriate signs of the background metric coefficients $\overline{h}_{\mu\nu}$. Indeed, the relation (11) contains $\mathcal{F}$, and the first of the relations (20) contains $\mathcal{I}$. One can check that this property holds at higher orders. Hence, the superluminal propagation reappears at higher orders unless at least one fine tuning relation is imposed at each order in $(g_{\mu\nu} - \eta_{\mu\nu})$ (in fact, our fourth-order analysis shows that the number of fine tuning relations is larger than one).

There is no obvious symmetry behind the relations like (11) and (20). Thus, helicity-1 gravitons propagate superluminally in the Fierz–Pauli massive gravity unless this theory is heavily fine tuned.

This work has been supported in part by Russian Foundation for Basic Research, grant 08-02-00473.

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